Learning to understand, remember and apply maths – consequences for learning contents and learning methods in the secondary levels

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What students wish and imagine:

• impartial teachers who know how to explain

• to be taken seriously and (having) to learn something „useful“

• to get learning chances – tolerant dealing of mistakes and clear orientations

• harmonious learning environment and fair assessment

1. What is the essential to be understood, remembered and applied in maths?

Subjects for network learning

versus teaching to the test

2. How can maths be learned in a way that the contents are understood, remembered and applied?

Specifying action competences

Up-to-date methods based on approved teaching and learning concepts

Different learning goals – different teaching-learning methods

Intelligent knowledge	Systematic, cumulative knowledge acquisition	Teacher-guided direct instruction	Disciplinary issue competence Form-master competence Diagnostic and didactic competence
Action competences	Practice-related, experience-saturated, situated learning	Topic work	Transdisciplinary issue competence Didactic competence
Meta-competences	Reflexively processed knowledge acquisition on own learning and acting	Guided independent learning	Diagnostic competence Didactic competence

1.What should be

understood — Understanding mathematic subjects ... as a deductive world on its own

remembered and — Knowing how to solve problems (heuristic capabilities beyond maths)

applied — Perceiving and understanding of the world around ... in a specific way

cf. Three basic experiences concerning mathematics according to H. Winter 1995

in maths
by maths lessons?

Understand, remember and apply – what?

First approach:

Thought experiment: Who is faster?

A rowing boat on a lake is doing a distance steadily to and fro.

At the same time a similar boat starts on a river and is doing the same distance as the other - but upstream and downstream.

Learning goals – an example

1. For a 800 m-footrace there is a certain time target. An average circuit time t is determined. To gain lead the first round must be run faster by 10 sec as at average speed.

   How much time will be left for the second round?

800m-total time:	2 t	1st round:	t – 10 sec
		2nd round:	t + 10 sec

Mathematical description: Arithmetic mean

$$\frac{a+b}{2}$$

Learning goals – an example

2. A moneylender wants to achieve an average interest rate of 8% per year. He offers his customer to pay only 2% interest in the first year but 14% in the second year. The interests will become due together with repayment of the borrowed capital at the end of the second year.

Problem:

$$\left[1+\tfrac{2}{100}\right]\cdot\left[1+\tfrac{14}{100}\right]=1{,}1628 \qquad \sqrt{1{,}02\cdot 1{,}14}=1{,}0783$$

$$\left[1+\tfrac{8}{100}\right]^2=1{,}1664$$

Mathematical description: Geometric mean

$$\sqrt{a\cdot b}$$

Learning goals – an example

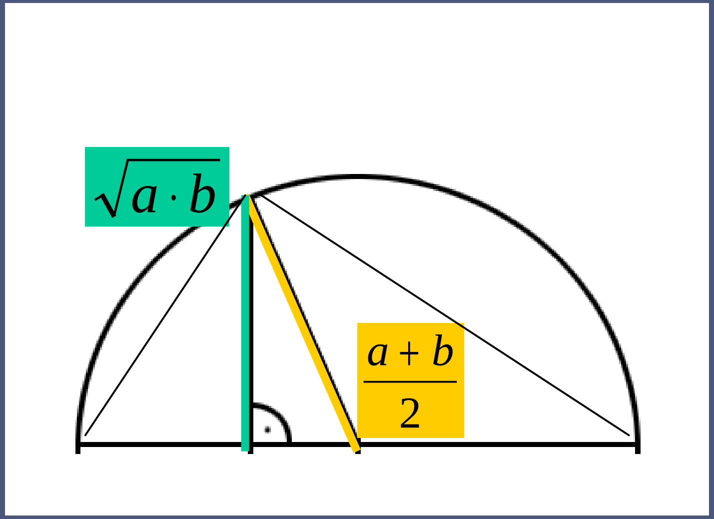
Observation: The arithmetic mean is slightly bigger than the geometric mean.

Questions: Is this always the case? And why?

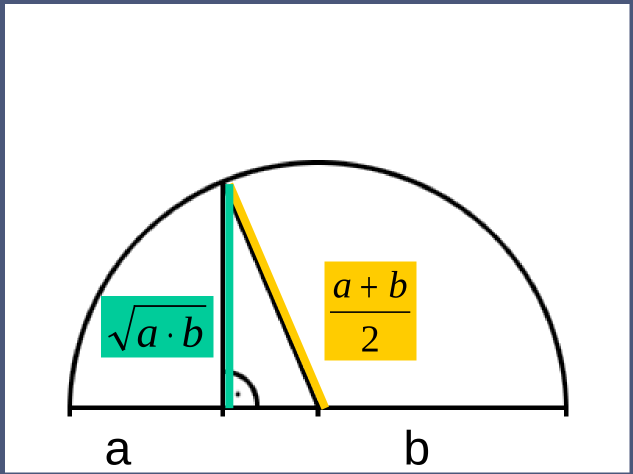
Mathematical level of description:

Assumption: $\frac{a+b}{2} > \sqrt{a \cdot b}$ a,b pos. reell

Justification by geometric interpretation:



Learning goals – an example



$$\frac{a+b}{2} \geq \sqrt{a \cdot b}$$

Extension: Is there an algebraic relation between arithmetic and geometric mean?

The pythagorean theorem allows the combination of arithmetic and geometric mean

$$\frac{a+b}{2} \cdot X = (\sqrt{a \cdot b})^2$$

$$X = \frac{2ab}{a+b}$$

Learning goals – an example

Arithmetic mean $\frac{a+b}{2}$

Geometric mean $\sqrt{a \cdot b}$

3. For a car trip to visit friends an average speed of 100km/h was planned.

Unfortunately there was a traffic jam so that half of the distance was done with an average of only 50km/h.

What should have been the average speed during the second half to reach the friends within the planned time?

Learning goals – an example

For the time at constant speed it is: $$t = \frac{s}{v}$$

Driving time 1st half + driving time 2nd half = total time

$$t_1 + t_2 = t_{gesamt}$$

$$\frac{\frac{s}{2}}{50} + \frac{\frac{s}{2}}{v} = \frac{s}{100}$$

$$\frac{s}{100} + \frac{s}{2v} = \frac{s}{100}$$

Interpretation: The planned time for the whole trip has already expired after the 1st half of the distance!

Learning goals – an example

What about this „mean“ in detail?

For the speed it is: $$v = \frac{s}{t}$$

Therefore it is for the average speed during both halfs of the distance :

$$\bar{v} = \frac{s}{t_1 + t_2}$$

And with $t = \frac{s}{v}$

$$\bar{v} = \frac{s}{\frac{\frac{s}{2}}{v_1} + \frac{\frac{s}{2}}{v_2}}$$

Simplified result:

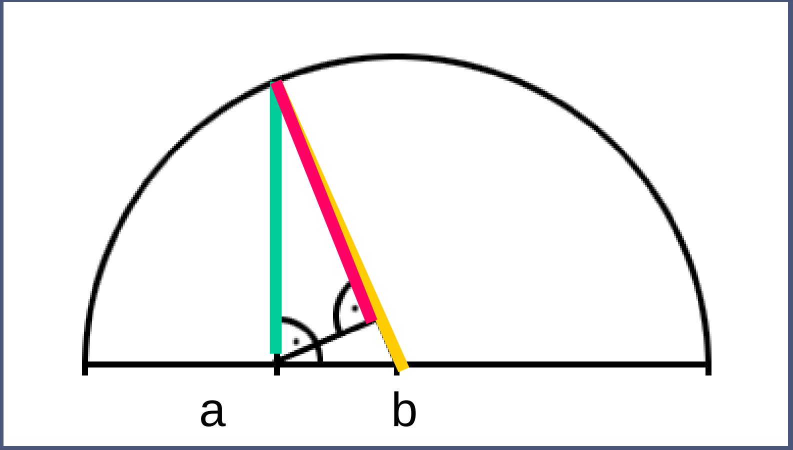
$$\bar{v} = \frac{2}{\frac{1}{v_1} + \frac{1}{v_2}}$$ Harmonic mean

Learning goals – an example: mean values in maths lessons

Questions: Where to find the harmonic mean, compared to the two other mean values?

Mathematical description level :

Transformation of term $\bar{v} = \dfrac{2}{\dfrac{1}{v_1} + \dfrac{1}{v_2}}$ into $\bar{v} = \dfrac{2}{\dfrac{v_2 + v_1}{v_1 \cdot v_2}} = \dfrac{2v_1v_2}{v_1 + v_2}$

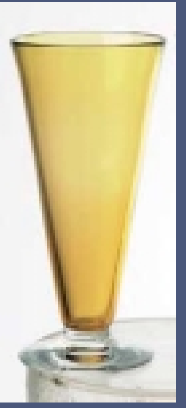


$$\frac{a+b}{2} \geq \sqrt{a \cdot b} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

Further means:

quadratic mean and cubic mean

$$\sqrt{\frac{a^2+b^2}{2}} \qquad \sqrt[3]{\frac{a^3+b^3}{2}}$$



With applications:

- Standard deviation

- When is a cup of wine half-full?

*Extension:

constructability of angle trisection

Learning goals – Consequences for the curriculum?

1.What should be

underststood

Function of maths to show structural differences in realistic situations

remembered and

The term mean value takes various forms

Example contexts and visualisations as reminders

applied
in maths
by maths lessons?

Mean values are understood as mathematical models which are remembered and used in different contexts

Reflection on procedures – adapations to support the learning effect

Subject "mean values"

Method for processing:

A closed starter problem is gradually enlarged, completed – thus ***opened:***

„Blossom model" *(PISA-problems)*

Expert method,
Internal differentiation with optional problems

Alternatives or variations of open problems for internal differentiation:

-closed, but *many solutions possible*

-Baker-Smith-problem:

The Bakers go for a circle walk of 12 km, scheduled for 4 hours as they have two small children. One hour after they have started there is water dripping from Mr. Smith's ceiling. The Baker's washing-machine is defect!

Mr. Smith is running to inform the Bakers. He is doing 5km/h. When and where will he meet the Bakers?

Would you try to follow?

"Funnel model" – team work, project work -
based on labour divison e.g. for modelling:

* how long takes a water exchange in the swimming pool?

-** find a new – and fair ! – membership fee of a club!

A sports club has 3500 members with 2000 adolescents. Until now, the adolescents paid a 5 € monthly fee, the adults 7 €. The whole membership fee takings have to be increased to monthly 34.500 €.

How to fix the new fees?

- ** in the fairy-tale "the Frog Prince" the golden bowl had to fall down – are there alternatives so that the fairy-tale can go on?

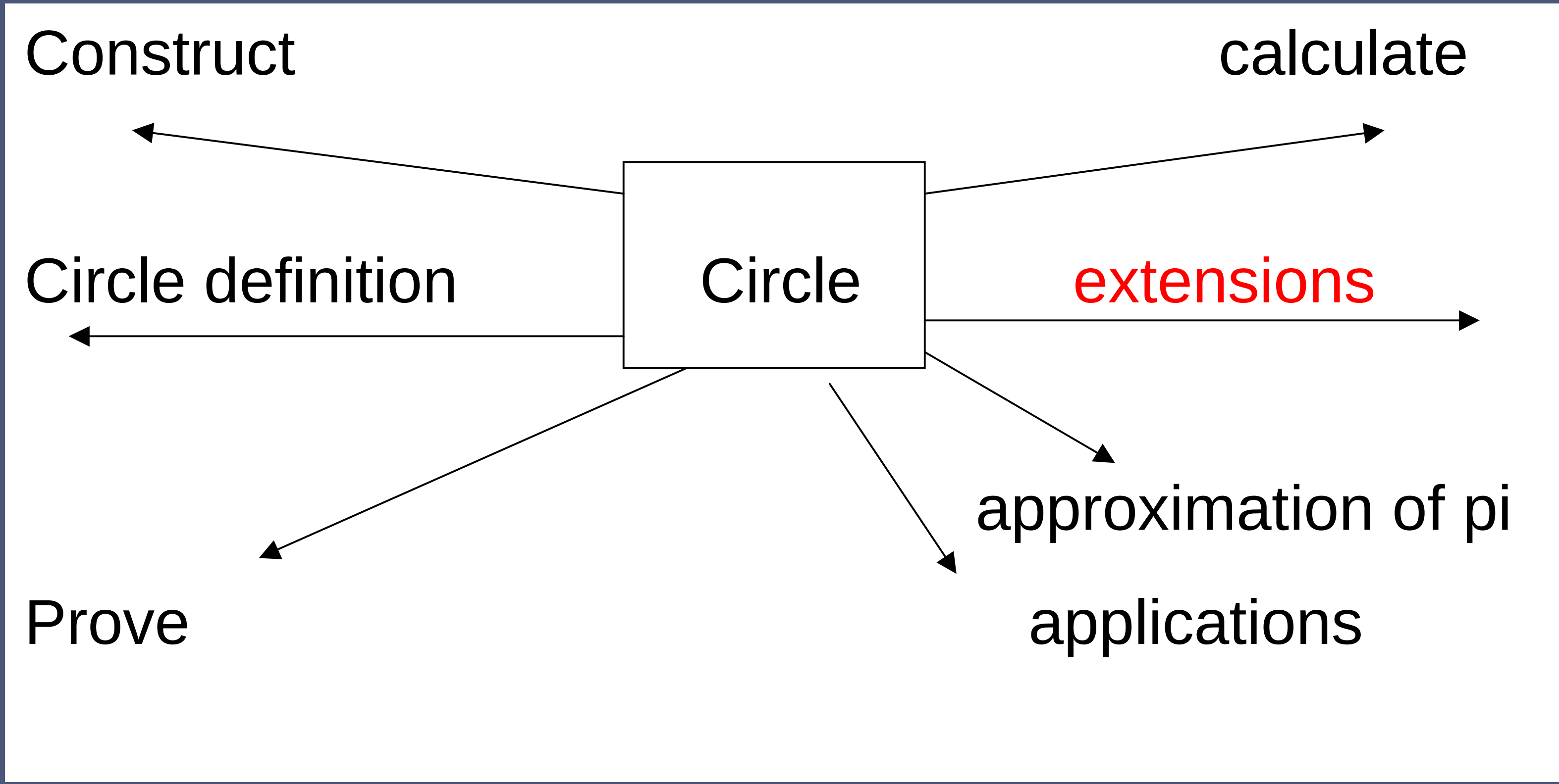
- *** modelling of a motorway exit

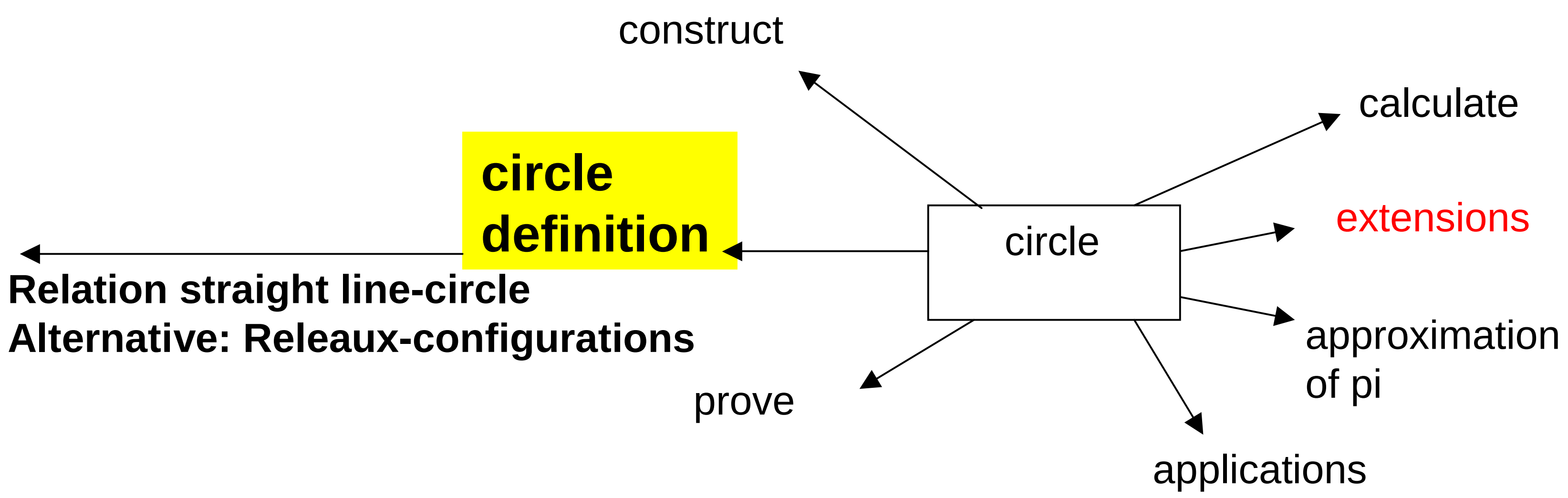
Subjects for network learning

Possible applications to support the curriculum spiral

- Handling money...
- Describing and comparing shares (fractions, rule of three, percentage, line partition/golden section...)
- ***Optimizing***
- Determining inaccessible points
- Describing assignments (growth/desintegration)
- Describing relations between numbers and figures
  - Visualisations (mean values...)
  - Congruence – similarity...
- Creating figures in plane and space
- Describing coincidences...

Prepare topic areas – choose main focus

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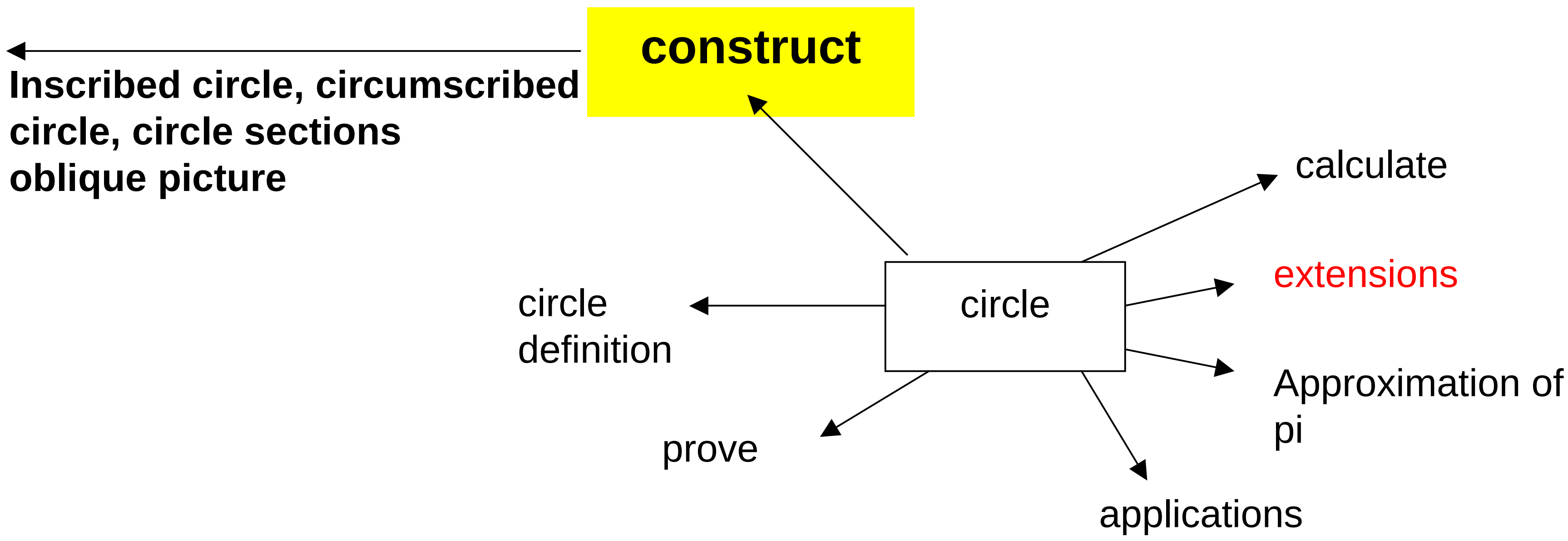
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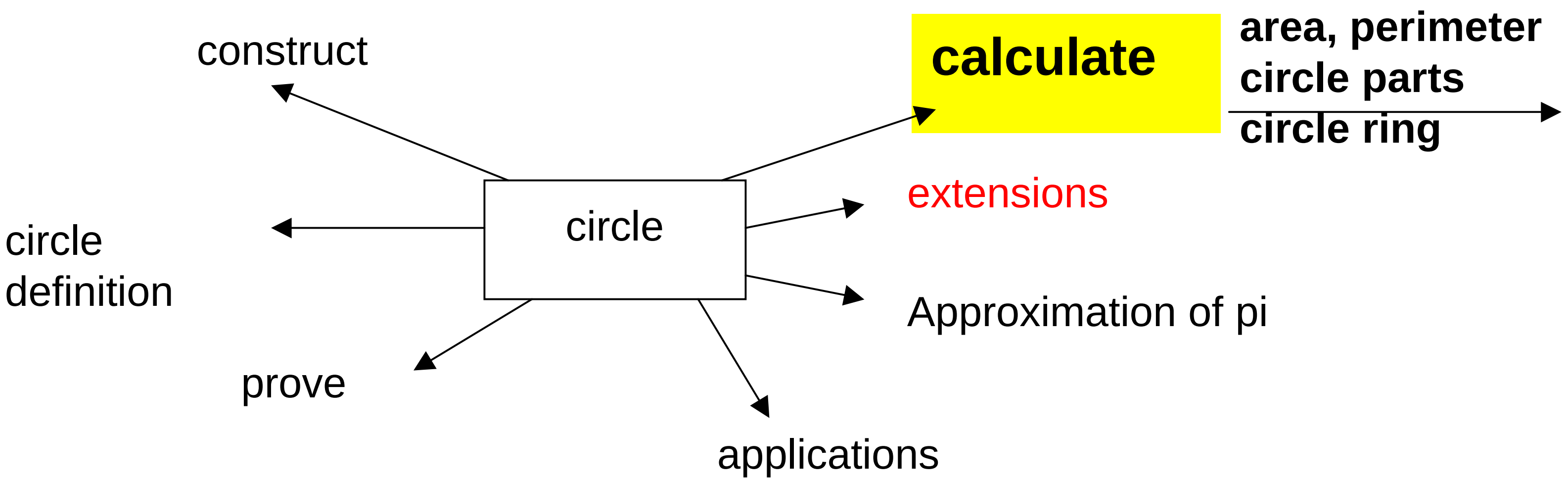
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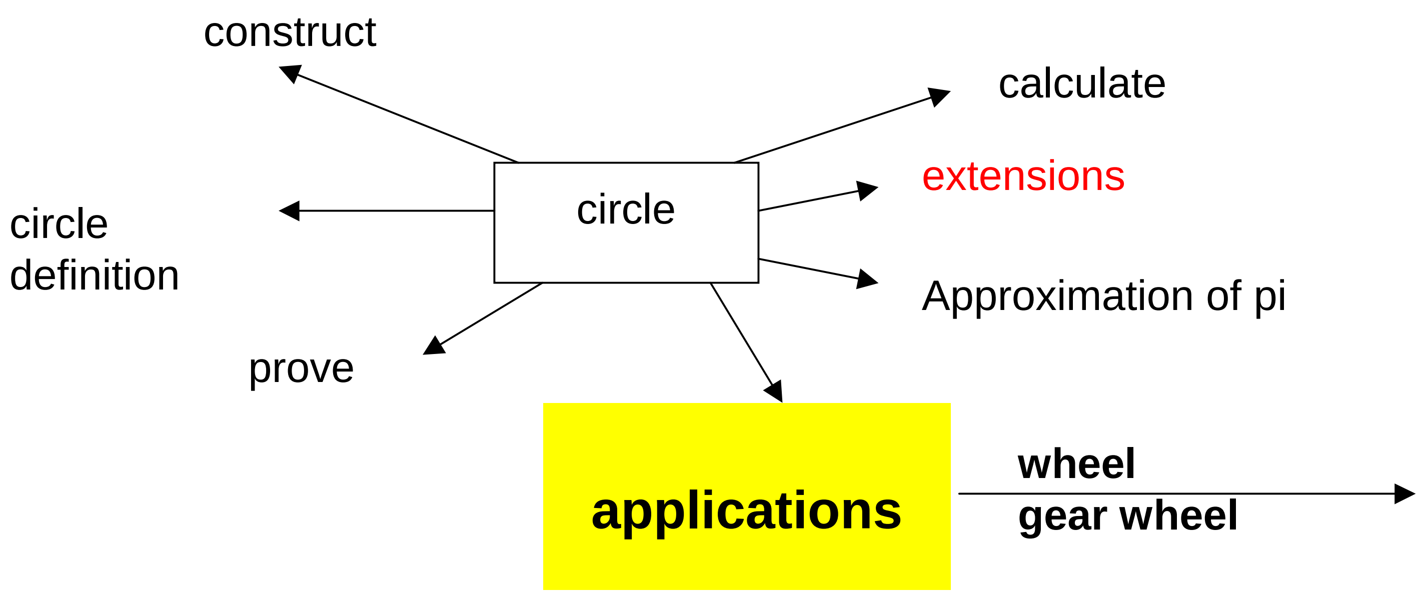
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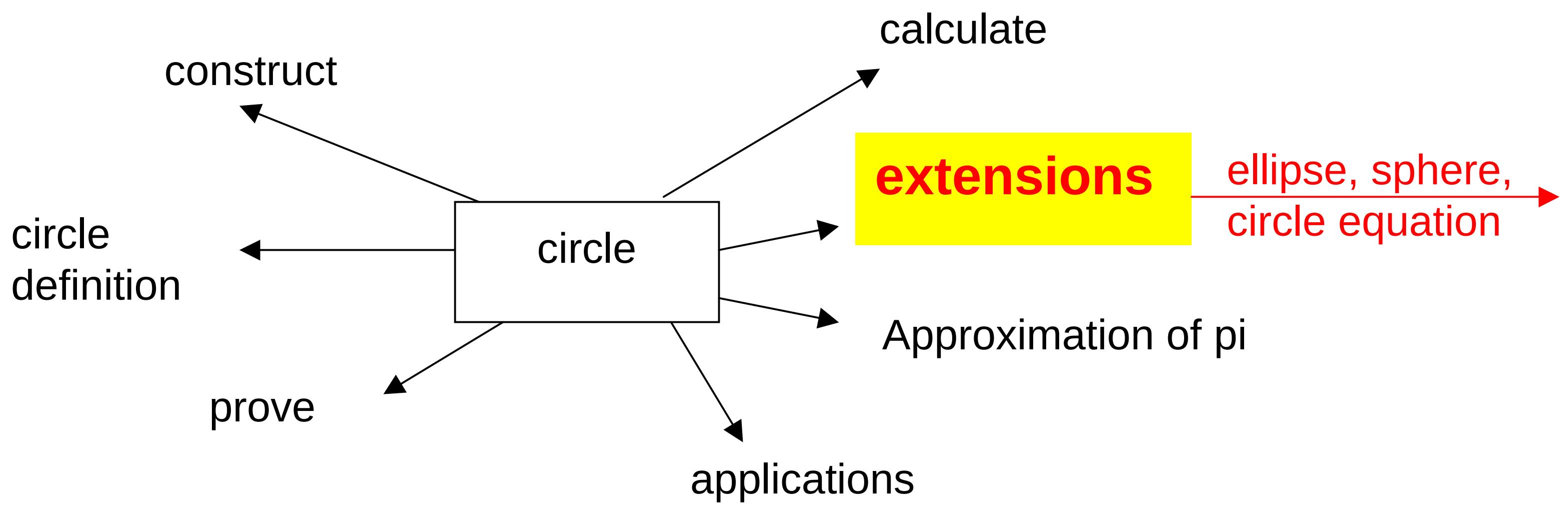
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1.What is the essential to be understood, remembered and applied in maths lessons?

Different subjects for network learning as learning offer

catalogue – related to fundamental ideas of maths and relevant application fields

filtering of main points by means of a semantic net (special analysis)

Adaptation with internal differentiation by open problem formats

Structure

2.How can maths be learned in a way that the contents are understood, remembered and applied?

Essential conditions to produce learning activities:

- Learning problems

requiring action: WHAT? WHY?

- Orientation basis for the required action

HOW can I proceed?

Problems – formats and types

Given	transfor-mations	searched	
X	X	X	solved problem (is this right?)
X	X	-	simple determination problem
X	-	X	problem to be proved, game strategy
X	-	-	difficult determination problem
-	X	X	simple inverse problem
-	-	X	difficult inverse problem
-	X	-	request to invent a problem
(-)	-	(-)	open problem

To understand – to remember – to apply requires:

Clear targets: Make sure that the „given“ learning goals are in line with the learning tasks

Start level: Make sure that the students have a real chance to manage the learning task

One suitable teaching method: ***Learning report***

Example of a learning report (class 9):

1. How can the length of an inaccessible distance be determined if measuring tape and protractor are at hand? (*describe introductory example*)

2a) Set up two matching equations for the given intercept theorem figure! (*give a sketch)*

2b) Sketch an intercept theorem figure for which is:

$x : 20 = (x + 40) : 28$ (inverse problem)

3. Which mistakes can occur if intercept theorems are used for calculations ?

4. When can intercept theorems be used and when not? Give examples!

Student‘s view:

1. Guideline for everything relevant (focussing)

2. Evaluation of the own learning level without assignment pressure

Teacher‘s view:

3. Clear position concerning long-term targets (also: safeguarding of start level and clarified understanding of targets)

Teaching method learning report

Arguments <u>in favour</u> of learning reports at the beginning or at the end of a lesson – without auxiliary tools:

- all students are addressed and integrated while time expenditure is low
- verbalisation of ideas
- comprehension deficits can be recognized and “repaired“ at an early stage

Recommendation

– Collect and comment the first learning report but don‘t appraise – discuss with the class and draw common conclusions...

Example of a learning report (class 11 - derivation):

1. (*Explain introductory example*)

2a) How can the path from the local alteration rate of a function to the slope in one point be described mathematically?

2b) What shows the derivative function of a function describing the filling level of a glass in proportion to time?

3. Which mistakes may occur when function derivations are determined?

4. In which cases can the derivation rule.... be used or not? Give an example for both !

   What is basically required to shape a derivative function? (existence, uniqueness)

Learning goals and learning chances in maths lessons:

Problem solving skills (heuristic capabilities beyond maths)

Realization method:

Targets approach via part activities of problem solving

Reference: mathematik lehren 115, Mathe-Welt

Part activities of problem solving in lessons and tests

The students

- recognize mathematical questions
also in daily life situations and know how to pose such questions

- City tour from a mathematical point of view ...
- Creation of new sweets or a tent...- will maths play a role?
- Where and how do we generally need to structure, to combine, to optimize, to justify decisions, to generalize, to interpret...

In a test:

Present a situation – ask to pose two questions requiring maths

Part activities of problem solving in lessons and tests

The students

- can communicate their ideas and give reasons for their work results (orally and in writing)

-Expert method for the handling of new learning subjects - student presentations

-- Learning report

- Maths stories

- Learning diary

- Portfolio for assessment of performance

Create occasions to reflect:

-Methods of self control as part of the homework

- write SMS with learning tips: What could help to understand a difficult problem?

Understand, remember and apply – modern and traditional teaching methods

- modular work scheduling – semantic net (mind map)
- tasks concept for one lesson unit with open questions for internal differentiation

  "blossom model", "funnel model", different solution possibilities
- permanent repetition (mental exercise, maths "driver's licence", knowledge storage)
- training of mathematical methods (heuristics) and self-guided learning

  reflection of methods (learning reports)
- training of part activities (for problem solving and linguistic-logical education)

Thank you for your attention!

www.math-learning.com   presentations, tips...

www.mathe-zirkel.de   student portal from class 7

www.madaba.de   problem data base for teachers

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