# Learning Methods and Techniques on How to Solve Problems

Regina Bruder

January 2003

**Prof. Dr. Regina Bruder**
TU Darmstadt, Department of Mathematics
Research Group Pedagogy of Mathematics

January 2003

# Learning Methods and Techniques on How to Solve Problems

Structure

# Learning Methods and Techniques on How to Solve Problems

## 1. Learning how to solve problems in mathematics lessons – a scope for everybody?

If you ask for the reasons that qualify a math lesson as educational and therefore accessible for everybody then “problem solving” obtains a fundamental role in the argumentation. According to Winter (1995) the following three basic experiences describe the educational claim of math teaching:

- recognize and understand phenomena in the world around us in a specific way
- perceive and understand mathematical subjects … as a deductive world on its own
- obtain the capability to solve problems through exercises (heuristic capabilities beyond mathematics)

Winter also describes mathematical problem solving as experience gathering for the own thinking capabilities. This intention can be illustrated by the apple tree problem, well-known from PISA-2000: Apple trees have to be planted in squares and surrounded by conifers as a shelter against the wind. Central subject are the functional relations between pattern number and number of apple trees (in squares), between pattern number and number of conifers (linear relation) and between the measurement of the square scheme (apple trees) and the number of surrounding conifers.

To understand the educational purpose and the correlations it is necessary to give the information new structures. The perception of functional relations between the “inside” and the “outside” is also relevant. Such relations between “content and brim” respectively “filling and cover” Winter is talking about in his lectures, lead to central mathematical problems like the isoperimetric problem (circle as biggest surface with given perimeter, analogous to the three-dimensional sphere) on the one hand and on the other hand to the considerable practical significance of the solution to these problems. With the help of mathematical perception the comprehension of daily life phenomena will be much easier, e.g. an elephant whose shape equals a sphere much more than most people needs his big ears to increase the surface of his body because his body temperature is regulated by the skin.
Although the apple tree problem is far from practice – as no gardener would have such ideas – it contains a big potential to experience ways of thinking like the formulation of central questions and their transfer to daily life as well as the application of typical mathematical working methods which will make this transfer to daily life possible. In this respect problem solving is – as well for the inner as for the outer mathematical context – a valuable learning goal of general educational character for all students.

With reference to Polya*(1965) maths teaching is often expected to develop heurisms of problem solving, connected with the hope that general abilities are created which can serve as a basis for life-long learning beyond school. In our apple tree example heuristic tools like classification tables and working with variables (using terms) would help to understand and solve the problem.

*George Polya, 1887-1985, was an Hungarian-American mathematician teaching at the ETH Zurich, worked especially on the field of problem solving.

Such tools are especially helpful to students with learning difficulties as they give if not a guarantee but an orientation to master unusual learning situations.*

As far as the results of the OECD-studies TIMSS and PISA allow any conclusions, we still seem to be far away from those general abilities of our students. Special difficulties arise from problems which require conceptual understanding and/or a flexible application of knowledge, cf. Baeumert and others (1927).

The English-speaking countries are classifying those learning requirements as “problem solving”. With respect to the given maths problems in contests (Math-Kangaroo, Math-Olympiad, Federal Mathematics Competition) this certainly must get used to. However, it might possibly demystify the problem solving and, especially after the alarming inequality of opportunities shown in the PISA-study, open the way to ambitious but affordable learning requirements for all students and to allow the experience gathering by own thinking according to Winter.
First it should be concretised which objectives under the label “learning of problem solving” should be pursued (on a realistic basis) in maths teaching for everybody (chapter 2) and what kind of circumstances are helpful (chapter 3) to reach them. In chapter 4 it will be explained which specific experience in problem solving and which knowledge of heuristic strategies and tools are necessary and how they can be won or obtained. Finally a long-term concept for math teaching, where different methods and techniques to learn problem solving have their specific places, will be presented, cf. chapter 5.

## 2. Which targets are pursued with problem solving studies in maths lessons?

### 2.1 Experiencing Eureka-effects

It is a wonderful moment when you can say: I managed it! A difficult problem, perhaps in your everyday life, or maybe a riddle is solved. You lean back and are deeply satisfied with yourself.

We know it was Archimedes** who was the first to say “Eureka – I have found it” in this context. Such an experience – although the individual problem solving will not necessarily lead to the Nobel price nor cost the life as it costs the goldsmith’s life in the example of Archimedes – still has another side-effect: The next problem will be tackled with more confidence, the own positive problem solving experience kept in mind – so why shouldn’t it be possible to master a new problem!

*In a study with more than 300 students of class 8 it could be demonstrated that the learning of heuristic elements leads to improved test results in maths and, especially, that the big number of those who are normally refusing difficult exercises, could be reduced by nearly 50%, cf. the examples and results in Bruder/Perels/Gürtler/Schmitz (2002) and the description of the whole study in Gürtler/Perels/Schmitz/Bruder (2002).

**Archimedes - 287-212 BC, Greek mathematician, physicist, and inventor. Few facts of his life are known, but tradition has made at least two stories famous. In one story, he was asked by Hiero II to determine whether a crown was pure gold or was alloyed with silver. Archimedes was perplexed, until one day, observing the overflow of water in his bath, he suddenly realized that since gold is more dense (i.e., has more weight per volume) than silver, a given weight of gold represents a smaller volume than an equal weight of silver and that a given weight of gold would therefore displace less water than an equal weight of silver. Delighted at his discovery, he ran home without his clothes, shouting “Eureka,” which means “I have found it.” He found that Hiero's crown displaced more water than an equal weight of gold, thus showing that the crown had been alloyed with silver (or another metal less dense than gold). – However, the poor goldsmith of the crown was killed.

To permit our students to experience such Eureka-effects, they must be confronted with suitable problems, which means problems of individual difficulty where obstacles have to be surmounted. At the same time, they must be offered special learning tools to surmount these obstacles – especially strategies for problem solving, so-called heurisms.

In the following some problems from different school and daily-life topics are presented. They will serve as examples to explain the heurisms afterwards.

***Motion problems***

1.1 “Average speed”

An average speed of 100 km/h was planned for a car drive to visit friends. Unfortunately there was a traffic jam and the average speed during the first half of the journey was only 50km/h. What should have been the average speed during the second half to reach the destination as scheduled?

*1.2 “Baker-Smith”*

The Bakers go for a circle walk in the mountains, scheduled for 4 hours as they have two little children. They start at 14 o’clock. One hour later there’s water coming through Mr. Smith’s ceiling. The washing-machine of the Bakers is defective! Mr. Smith is running to inform the Bakers. He is doing 5 km/h on average. When and where could he possibly meet the Bakers? Would you follow him?

*1.3 “Circle meeting”*

Two volumes are moving towards on a perimeter. One of them puts 3 minutes to circulate, the other 5 minutes. How much time is there between two encounters? Where do the encounters take place? Which part of the perimeter will be passed by each volume between two encounters?

*1.4 “Bathing cap”*

When Hans jumps from a bridge to take a bath he loses his bathing cap. Only after 10 minutes of upstream swimming he recognizes the loss. Turning round he follows the cap, catching up with it 1km off the bridge. What is the speed of the water running in the river?

***2. Inequalities***

2.1 “Root inequality”

It has to be demonstrated that for positive real a,b,c,d it is:

$$\sqrt{(a+b)(c+d)} \geq \sqrt{ac} + \sqrt{bd}$$

2.2 “Fraction inequality”

It has to be demonstrated that for positive real a,b,c,d it is:

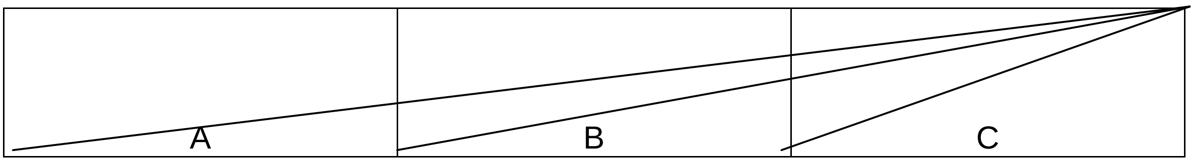
$$\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c} > \frac{3}{a+b+c}$$

### 3. Optimizing

“Packaging”
Find out mathematically optimized packaging forms according to a defined volume (prism, cylinder, cone, pyramid) for daily life!
Explain why the mathematically minimizing dimensions for packaging e.g. for sweets packaging are not often used.

### 4. Geometry

“Neighbouring squares”
What is the relation between the angles at A, B and C?




### 5. Age

“Father and me”
When my father was 31 I was 8 years old. Now my father is double my age. How old am I today?

### 6. Order of numbers

“Remembering telephone numbers”

Find a simple scheme to remember the following two telephone numbers:

29 16 23 and 30 37 44

Dealing with these problems you will see that the examples may have different effects on you, perhaps you will solve one problem somewhat faster and maybe another problem will take a bit longer. Other persons might react in a different way and the ideas of solution may be completely different, depending on the situation and mental state when someone is confronted with the problem, on the individual knowledge and experiences in problem solving as well as on both intuition and the personal way of thinking.

Basically the terms “problem” and “problem solving” cannot be defined without the addressee of a problem. If the target is to teach problem solving, different degrees of exercises must be available to allow all pupils to take advantage from it. We refer to the kind of exercises that are of central importance to the lessons as a general concept, cf. Bruder (2000d). “Problems” are specific exercises which are confronting the addressee with special difficulties. An exercise becomes a problem when the requirements are – or seem to be – unfamiliar and do not permit schematic working. The defining lines are fluid.

Intuitive and successful problem solvers often show high intellectual flexibility (transfer capability) on a specific field of knowledge.
Intellectual flexibility is a thinking procedure just like conscious, methodical, independent and active thinking, cf. Lompscher (19972). Such thinking procedures depend on a context and can be trained, i.e. a lack of mental flexibility concerning mathematics can be “compensated” up to a certain degree! Cf. chapter 4.

## 2.2 Realistic objectives for problem solving studies in maths lessons

What could be realistic objectives for the learning of problem solving in maths, taking into consideration the above statements?

The students

- recognise mathematical problems also in daily life and are able to formulate such problems

- are aware of mathematical models (pattern of mathematification) respectively suitable proceedings (heurisms) to work on mathematical problems and they are able to use them according to the situation

  beginning with information gathering to partial solving and ending up with the discussion and presentation of the results

- develop willingness and reflection capabilities for their own activities.

A possibility how this aim can be transferred to the pupils is shown in the following chart (class 7/8)

---

**Mathematical problem solving – what can we learn from it?**

- **Tools – for example: Equation, table, informative figure**

**and**

- **Strategies – for example: Backward and forward working, analysing**

**to solve a problem and to find ways of flexible thinking!**

**This requires above all:**

- **Training of question-asking and –formulating**
- **Observing oneself when a problem is solved**
- **Assessing oneself: What I’m good at? What I’m less good at?**
- **Experiencing different solution possibilities to develop own preferences and skills and learning to think into different directions**

---

### 2.3 Asking questions from the mathematical point of view – central part of problem solving

Problem solving means formulating questions – but what kind of questions?

Problems which aren't understood, can't be solved. The question is:

- **What is it about?**

Successful problem solving requires solid basic knowledge. Therefore the following question may help:

- **What do I know in connection with the problem?**

Problem solving has a very experimental character – and requires also "trying". With a strategy this "trying" will lead to a success rather than working only with "trial and error".

- **Which methods and techniques do I have at my disposal – will they help to solve the problem?**

Finding own questions and not only answering determined questions by the teacher is connected with individual learning, with development of creativity and above all with the skill to solve problems. Wanting to solve a problem means to always formulate the "right" questions.
It is the flexible point of view on our world of thoughts and ideas which leads to new questions – it is also the changing point of view which is so essential for successful human communication. The (heuristic) formulating of questions can be learned – here with respect to mathematics and mathematical applications – by exercising the fundamental ideas of mathematics and the typical mathematizing procedures, like the forming of concepts or demonstrating and proving.
The mathematical way of thinking can only be understood after having learned to ask basic questions from the mathematical point of view, as for example:

> *How can the given situation be structured?*
> *What is the nature of the correlations which are to be described mathematically?*
> *Can there be a clear solution to this problem?*
> *Are there easier alternatives?*

The often administered "open problems" – open with respect to a solution way and/or a defined problem – have the charm that they confront the students with the necessity to formulate suitable questions. Nevertheless they do not explicitly teach the formulating of questions which can only be acquired indirectly. The openness of this type of exercise also expects too much from less capable students. This can be compensated by training, without adopting a scheme. The strategy of forward working permits not only to solve calculation- and construction problems but also to find interesting questions.

In a problem solving training in class 8 (applicable from class 5 and almost unchanged also in the upper classes) we asked everybody to write down a maximum of different possibilities to use a brick. Those who were able to find ten really differing applications within one minute are intellectually very flexible (on this field).

Afterwards the different ideas were presented to give everybody an overview about the multitude of application possibilities. Then a strategy was presented to find out as many applications as possible: the brick is examined with respect to its characteristics which are noted as follows: shape, material properties and weight that determine – now more systematically – various application possibilities. Cf. Math-World-Insert in “mathematik lehren” (“maths teaching”), issue 115 from Ables (2002).
In a second step the pupils transferred this method of systematic finding of application possibilities on the basis of object properties to other given objects – e.g. a cup, a pencil or a piece of cardboard. The effect was that this strategy which we call “forward working” clearly helped to find more application possibilities than without it before. Such a measurable success encourages and gives confidence in the own “trained” creativity and allows to recognize and understand the usefulness but also the limits of heuristic strategies.

Strategy:

*Forward working*

What are  the given facts?
What do I know about the given facts?

⇨ How can I make use of it?

Under didactic aspects the effect of heuristic strategies can be described in a few words:

> *If it is possible to develop subconscious problem solving methods of intellectually flexible persons and to train and use them consciously in the form of heurisms it is possible that they will work successfully like intuitive problem solvers.*

To avoid unattainable expectations it must be underlined: Heuristic strategies give impetus to further thinking but they do not guarantee a solution like an algorithm.

After discussing in the lessons the usefulness of strategic knowledge this essential fact should be transferred to mathematical situations, e.g.:

> *Imagine you are working in the development department of a company and have to create a new sweets specialty. Which questions requiring mathematical answers must be considered for your new development?*

The students must be aware that question-asking is “only” part of the mathematical problem solving.
However, a relation to everyday life is already recognizable: in which situations can it be useful to know or find a maximum of different possibilities? These could be decisions where more than one option would be welcome, or dilemmas where a solution is needed.

*Exercise and use of forward working*

Formulate **mathematical** questions which might be of interest for one of the following situations. Try to use the **strategy of forward working**!

a) You are working in a company that produces orange juice in tetra-paks
b) You are working for Ferrero in the Hanuta department
c) You are helping at home to renovate the bathing room
d) You want to make a candle
e) You are planning a swimming pool for your garden.

My mathematical questions are…

We still haven't solved a sense-making "real" mathematical problem with our application possibilities for different objects or with the formulated questions. But we learn to ask mathematical questions and to precise a complex problem which will generally be easier to solve.

However, the capability to ask oneself questions has another, deeper importance. According to Polya (1981) it is useful to find heuristic strategies for the problem solving in the form of questions.

Polya divided problem solving proceedings into four phases and developed a manual for problem solvers. This manual is a set of questions intended to give the problem solver clues for further thinking possibilities at every phase of the problem. If the instructions are followed, says Polya, the chances are good to solve the problem.
Below an overview about the summarized instructions:

**1st Phase: Understanding of the problem**
**What is unknown? What is given? What is required?**
Is it possible to fulfil the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or over-defined? Or contradictory?
Draw a figure! Find a suitable name!
Split up the different parts of the condition! Can you write them down?

Polya underlines the importance that first of all the problem solvers have to get clear about the type of problem. In this phase it is not yet necessary to make up a plan – or to start a solution. As the PISA-study demonstrates the comprehension of the text has a decisive influence on the ability to solve a problem.

In the second phase Polya presents a number of questions which can certainly help to systematically develop a target-oriented plan. However, modern problem solving prefers single strategies to a question complex, cf. the “general model for the problem solving”.

2nd Phase: Making a plan

- Have you already seen this problem? Maybe in a slightly different form?
- Do you know a similar problem? Do you know a theorem that could help?
- Look at the unknown! Try to remember a problem with the same or a similar unknown.
- Here is a problem like yours, but already solved. Can you make use of it? Can you make use of its result? Can you make use of its method? Would you introduce some tool so that you can make use of it?
- Can you put the problem into other words? Can you formulate it differently? Find a definition!
- If you can’t solve the present problem, try to solve a similar problem first. Can you imagine a similar problem where an approach is easier? A more general problem? A more special problem? An analogous problem? Can you solve part of the problem? If you keep just a part of the condition and leave out the other, how far is the unknown defined and how can I modify it? Can you find some help in the new data? Can you imagine other data which could determine the unknown? Can you change the unknown or the data or, if possible, both of them so that the new unknown and the new data are closer together?
- Did you use all data? Did you make use of the whole condition? Did you consider all essential terms included in the problem?

**3rd Phase: Executing the plan**

- If you proceed according to a solution plan, **control every step**. Are you sure that the step is right? Can you prove it is right?

**4th Phase: Review**

- Can you **control the result**? Can you control the proof?
- Can you deduce the proof in different ways?
- Can you use the result or method for another exercise?

Polya formulates the forth phase for the following two reasons: As a control in the traditional sense to make sure that the solution of the problem is correct.
On the other hand the results from the solution proceeding shall be used also for other, future problems.

These questions can of course not simply be presented to the students with the intention that they are learned by heart. According to a constructive learning concept they have to be elaborated and – if possible also formulated – on the basis of the pupils’ own experiences.

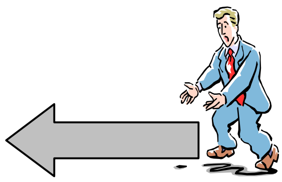
Suitable tools and strategies are discussed from *model exercises* and trained with *optional exercises.*

The following example can also be used as a model exercise for backwards working from class 5/6:

*"The  seven gates"*
A man picks up apples. On his way into town he has to go through seven gates. There is a guardian at each gate who claims half of his apples and one apple extra. In the end the man has just one apple left. How many apples did he have first?

This problem can be easily solved by making backward conclusions, adding a suitable presentation (vector diagram, Venn diagram, table). Finally the students make proposals how to visualize the strategy and formulate suitable auxiliary questions comparable to the forward working:

| | |
|---|---|
| Backward working: | What is searched?<br>What do I know about the searched?<br>What do I need to find the searched? |



**Orienting questions to support the use of heurisms in problem solving (general model for the problem solving):**

- What is the problem about? (own description of the problem)
- How can the problem, by means of current terms,  be put in a more comprehensive or even simpler form?
- How can the problem be visualized or presented differently *(heuristic tools)?*
- Which similar problems were already solved? How? (*principle of analogy*)
- In which parts can the problem be split up? (*principle of  reduction*)
- Can parts of the problem be reduced to already solved problems? (*Reduction of the unknown to the known)*
- What kind of exercise does the problem represent? (possible use of *special heuristic principles)*
- What can be concluded from the given details? (*forward working*)
- What is needed to find the searched? (*backward working*)

Questions for the reflection phase:

- Which solution strategies helped to solve the problem?
- Which new proceedings became apparent with  the solved problem?
- Which solution way was best for the problem?

Our teaching experiences and results from a research project on the learning of problem solving show that heuristic strategies and principles can be learned by suitable formulating of questions which leads to increased achievement and clearly decreased refusal in front of difficult exercises, cf. (among others) Bruder/Perels/Gürtler/Schmitz (2002).

Details for the adoption of Polya's ideas in the math lessons, summarized in our "general model for the problem solving" (see above), are included in chapter 5, the heurisms are defined in chapter 4.

## 3. Conditions and requirements for the learning of problem solving with lasting effects

### 3.1 Clear targets

For the learning success it is of decisive importance which problem the students deduct from a given exercise and how this subjective problem – we call it *personal exercise*– is going to be tackled.

In the lessons it can be observed that many students tackle a given exercise successfully by asking and trying to understand the intention of the teacher. But it can also be observed that the individual transformation of the given exercise into a personal exercise fails partly or completely. Instead of working consciously on a question, a movement or a text, some students are only working mechanically on given facts or they are working on something completely different. In this case the given exercise does not correspond to the self-constructed exercise. A correspondence would however be important for the individual learning success.

If the students could be induced to take over more responsibility for their own learning activities and to find clear targets and a personal conception of the learning contents many exercises would be tackled less mechanically and with more inner participation.

These three aspects are prerequisite for the transformation of exercises into a personal exercise. If, in addition to this, it is possible to make students think about their method to solve a problem, the chances are good that by means of such reflected problem solving they will have learned something lasting and weren't just "busy".

Therefore it is not so important to find "top" exercises as finally the way how the exercise is dealt with is decisive for the learning success. Time and again it can be observed that successful teachers are able to "make something" out of a supposedly trivial exercise. In the three phases of an exercise – the gathering of information together with the creation of a personal exercise, the processing of information and the presentation of the result– the pupils can be provided with precious orientation aid by a specific handling of the exercise and thus with own thinking experiences.

There is no automatism and it depends on many factors how suitable personal exercises for students can be initiated. In this context we would like to mention that the results of the empiric studies by Jäger/Helmke (2001) as part of the project

MARKUS show that good learning achievements require a good “classroom management”. This will not surprise an experienced practising teacher. Therefore it is primordial for teachers to ensure positive learning conditions so that target-oriented learning exercises can be individually developed. Important is a calm, concentration- and analysis-stimulating atmosphere, especially in the phase of information gathering. The best exercises are useless if they aren’t successful – in many respects. A positive learning atmosphere is a necessary but still insufficient precondition for this.

## 3.2 Problem solving – wanted, allowed, possible !

The following learning requirements, which are development-adapted on the one hand and development-improving on the other hand, are, on the basis of corresponding targets and contents of the maths lessons in connection with a creativity-friendly learning atmosphere, suitable to produce creativity in maths lessons:

- It is allowed to make a mistake and a piece of rough paper is allowed, too (assessment-free moments during the lessons, time to think – possibility of stress-free learning)
- Every idea is taken seriously (mutual esteem of students and teachers; unprejudiced contacts)
- Different proceedings are not only tolerated and accepted but also adopted – for example in tests
- The students’ need for orientation is taken into account by clear targets and valuation standards as an orienting guideline for the personal learning and by avoiding small-step procedures which are rather obstructive to independent acting and thinking

Concerning the targets and issues of maths lessons it should be aimed at

- a balanced relation between formal and application-oriented competencies,
- coordinated preparation of the courses to avoid insular knowledge,
- teaching of different mathematical terms, correlations and procedures as well as of methods and techniques which can be used for one special topic (modular subject-oriented curriculum structure versus one-dimensional systematically theory-oriented structure) so that a large fund is available.

Concerning the nature of the learning requirements the following methodical aspects are of importance:

- ***optional exercises***

(to be chosen personally from an exercise fund on one topic with different degrees of difficulty and sometimes in different variations)

Prerequisite that these possibilities are made use of (avoidance of over- or under-challenging) is that the students are gradually enabled to come to a realistic self-assessment (realization of capabilities and weak points) and to take over responsibility for their personal learning.

- ***“open” exercises***

as learning requirements which can be managed **by different procedures** and **with different results** (e.g. find a story on a path-time-diagram; compare several exercises and their solutions under different aspects) and especially:

- ***invention of own exercises***

as learning requirements from which further, additional or more detailed questions can be developed by modifications/variations/supplements of a given exercise **(blossom model** of the variation of exercises - especially suitable for the inner-mathematical formulation of questions) in the form of problems where mathematical questions have to be developed first (funnel model of the variation of exercises – e.g. in application- or modelling exercises)

- ***problematic exercises*** in the “zone of the next development phase”, e.g.:

“River width”
Find as many different methods as possible to determine the width of a river with side marker, protractor and tape measure!

Remark: There are more than 10 different solutions to this problem. It can already be solved with the basic knowledge on angles and right-angled isosceles triangles available in classes 5 and 6. In the course of the school years the mathematic instruments to determine a distance are gradually increasing (parallelogram, intercept theorems, trigonometry, vector methods).

The following aspects are essential to make students **want** to learn problem solving:

- Praise and appreciation for particular achievements, for unusual, new solutions; support of sound competition
- The experience that maths can be generated, the connection between sense and significance becomes transparent
- Choice of subjects in dependence of the students’ age, consideration of their changing leisure interests and surroundings without completely orienting the subjects to them – students must also be confronted with facts they are not (yet) interested in
- The exercise contains some surprise and arouses astonishment, which leads to questions like: How is this possible? Is this always the case? Is this really right?
- The formulation of a question is addressed to the students’ supposed competences:

  *How would you decide?*
  *Discuss your decision!*
  *Is there another way – or another possibility?*

Another condition connected with motivational aspects consists in an adequate quality of the requirements:

If exercises in a concrete learning situation are **appropriate to and supporting the development** of a student, they can be assumed as a challenge.

The art of teaching consists for the most part in finding such adequate and thus motivating demands to take learning action and to assist the initiated process of dealing with it. A solution to this problem are the so-called “open exercises” which have differentiating effects as the students are allowed to choose the level of processing on their own. However, this procedure implies another problem: how can a desired learning success be reached (only!) by open questions?

We can state that: a multitude of exercises is not yet a guarantee for successful learning and: exercises which offer more room to construct individually suitable learning exercises improve the learning success. However, a certain quality level must be guaranteed.

## 4. Heuristic tools, principles and strategies

### 4.1 The effects of heuristic education

The proceedings of intellectually flexible problem solvers can be analysed and described, the results are heuristic principles, rules, strategies or tools. Polya, but also Engel, König, Sewerin (1979) and others formulated such heurisms. If it is possible to learn and flexibly use them, it is possible to reach similar effects to those generated by intuitive problem solvers. They are appropriate to support both capable and less capable students if they are helpful in cases where it was difficult to do without them. This means that the exercises have to be difficult enough but still soluble while aiming at “the zone of further development” (Wygotski).

The point is to learn suitable heurisms in the lessons and to use them unconsciously step by step. Special model exercises will take over the function of a mnemonic trick. Model exercises with a name are easier memorized.

Heurisms are learned like driving: at first the levers are operated with utmost concentration though a bit clumsily but soon the course of action (strategies, tools, principles) is managed without much thinking. While the different levers are already mastered unconsciously in well-known situations (no problem any more!) their function will be remembered in critical situations (problem!) and used if necessary.
Heurisms are intellectual tools and their flexible application must be learned. How can this be reached?

Successful problem solvers are **intellectually extremely flexible.** This can be seen in the following actions taken in mathematical problem solving, cf. Lompscher (1976) and Hasdorf (1976):

| | |
|---|---|
| Reduction: | The pupils reduce the problem correctly to its essential part, they are able to focus. |
| Reversibility: | The pupils are able to think backwards and do this in certain situations automatically. In daily life this capability is also needed, e.g. when searching a lost key or glasses. |

| | |
|---|---|
| Consideration of aspects: | They take several aspects of the problem into consideration simultaneously or recognize the correlation of facts and are easily able to vary them. |
| Change of aspects: | If needed they exchange assumptions and criteria to come to a solution. By intuition they consider different aspects of the problem which avoids or overcomes getting stuck |
| Transferring: | Good problem solvers are able to transfer a known proceeding to another, sometimes completely different context. They can see the point of the problem. |

Untrained talented problem solvers are generally not able to make use of these capabilities deliberately. Therefore they are often unable to explain how they solved a problem.

There is a multitude of school-relevant heuristic proceedings and techniques. A reasonable structure is presented by Lehmann (1990) as follows:

**Heuristic tools**

Table
Variable/equation

informative figure
solution graph

stored knowledge

**Heuristic strategies**

Forward working (FW)
Backward working (BW)

combined FW and BW
systematic trying

Search for equations, relations or mathematical patterns

**Special heuristic principles**

Invariance principle

Principle of definition by cases

Extremum principle

symmetry principle

decomposition principle

Drawer principle *

working with special cases

**General heuristic principles**

Principle of analogy

reduction principle

transformation principle

*The drawer principle is almost never used in normal school lessons which is regrettable. However, some sources on how to foster pupils interested in maths offer many comprehensible and stimulating applications.

Now a relation between the manifestations of intellectual flexibility and particular heurisms which support them is evident:

| **Aspect of flexibility** | **heurisms** | **partial activities** |
|---|---|---|
| **Reduction** | *informative figures* | illustrating (understanding the exercise, finding a simple model) |
| | *tables* | structuring the facts |
| | *equations* | describing correlations (modelling) |
| **Reversibility** | *backward working* | reversible exercises *What should I know to determine the unknown ?* |
| **Consideration of aspects** | *invariance principle* | *Look for common aspects in different facts !* (to grasp relativity or several aspects simultaneously) |

Examples:

Find the principle of numerical orders (searching invariants)
Meeting point exercises (*the place doesn't change!)*
Age exercises (*age difference doesn't change)*
Puzzle: looking for edges and borders

| | | |
|---|---|---|
| | *extremum principle:* | Searching solutions for boundary values or extreme cases among different possi-bilities! |
| | *Principle of decomposition:* | How can the formulation of the problem, the facts or the mathematical object be cleverly decomposed or fractionalized? |
| **Change of aspects** | *combined backwards and forward working* | |

Example:
Two metal cubes with a given integer edge length, e.g. 3cm and 4 cm, are melted together to a cuboid. Which integer dimensions could have such a cuboid?

| | | |
|---|---|---|
| | *Transformation principle* | transfer to a model level |

*Search for other mathematical ways to describe the given and the unknown!*

Vary the conditions! Examine the given and the unknown in different correlations!

Decompose, complete and connect with something new!

Examples:
Reflection on space problems, vector description of elementary geometric correlations

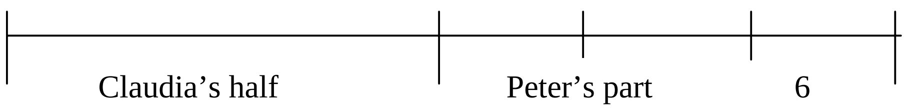
## 4.2. Heuristic tools

In our studies we found out that heuristic tools can easily be learned at secondary school level I, that they are very convincing and lead, after a short training period, to rapid success in the solution of difficult application exercises. Moreover heuristic tools are suitable to demonstrate intuitive solutions of capable students in a comprehensible way.

An example:

"Distributing marbles"
Claudia takes half of the marbles out of a sack and keeps them for herself. Then she passes two thirds of the remaining marbles to Peter. She has 6 marbles left. How many marbles were in the sack first?

The fact that a sixth of the marbles corresponds to the remaining marbles can be shown on a graph in a similar way to what happens in the head of very "flexible" problem solvers. The unknown total of marbles in the sack is symbolized by the length of a line. This line is first divided in two, one half is divided into three parts.




This shows that it is not absolutely necessary to put such an exercise in an equations context. Of course that means that in a test any other approaches in addition to the (possibly just learned) equation method should be equally assessed, cf. Bruder/Weigand (2001).

### 4.2.1. The informative figure

The tip to make a sketch first is more than known. What is meant is that first an idea of the exercise context should be developed. However, this tip is sometimes not sense-making as many students have not learned to make a helpful sketch.

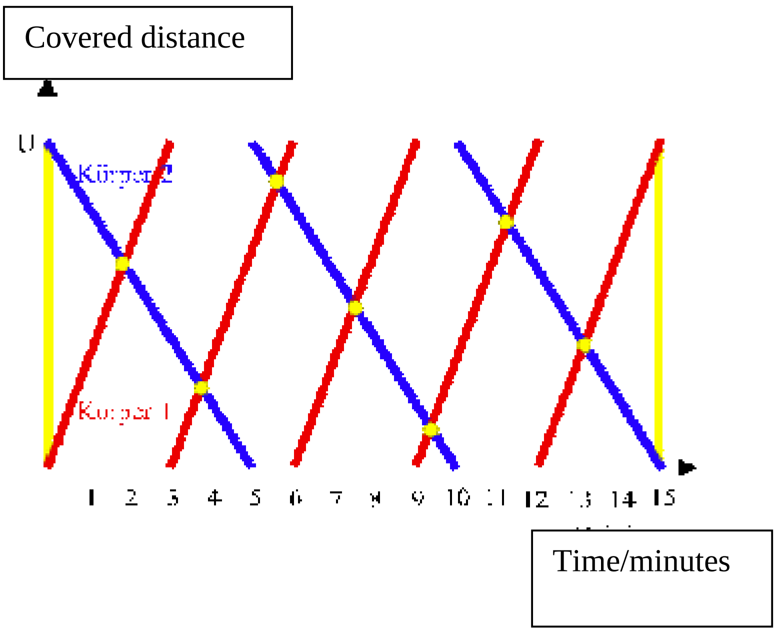
In geometry it is often helpful to relate the given and the searched parts to something known, e.g. a (right-angled) triangle. We then call a “helpful sketch” an informative figure. The visualization of algebraic relations plays an important role.

However, such relations must be explicitly learned to reach that the product of two numbers can be visualized as square surface and a certain number as length of a line. The coordinate system is added to represent classifications.

For motion- and age problems there are some nice geometric or graphical solutions with orienting effects for similar examples even if they were not found out by oneself but presented by others.

*Solution “circle meeting”*
The sketch shows that the volumes meet 8 times in 15 minutes. The encounters take place within equal periods, every 15/8 minutes, amounting to 112,5 sec.

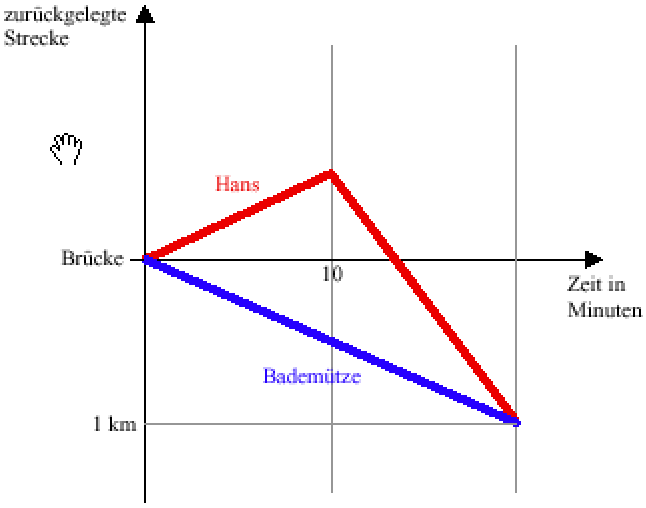
The meeting points divide the circle into 8 equal parts. So the faster volume crosses 5/8 of the circle, the slower 3/8.

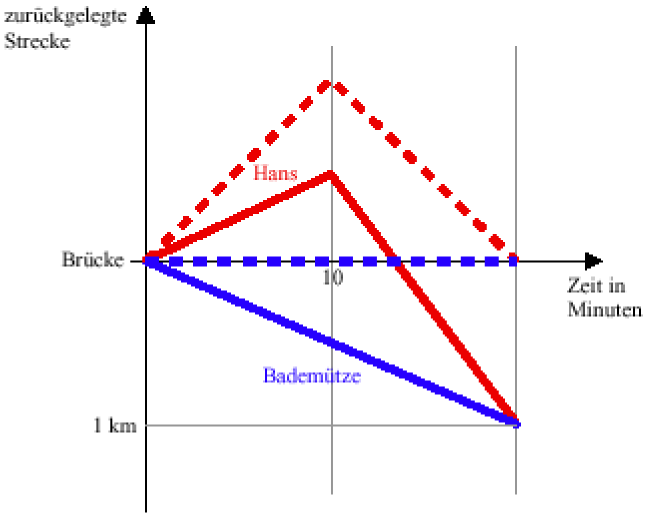


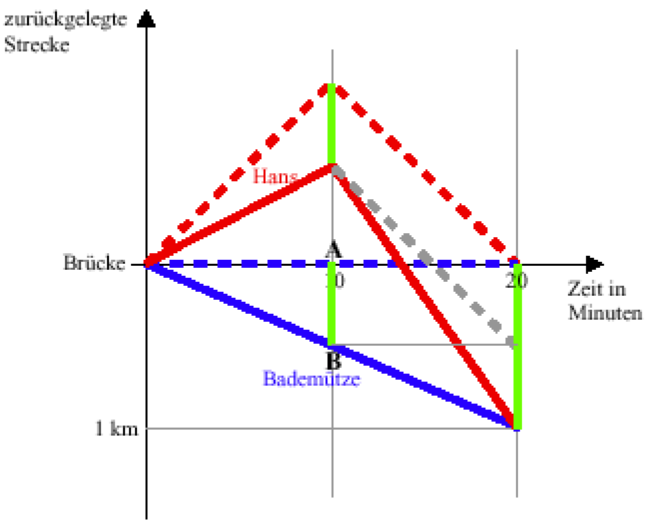

Below the graphical solution of the bathing cap problem – the problem was taken from a school book of class 9 (Klett)

Covered distance

Time/minutes

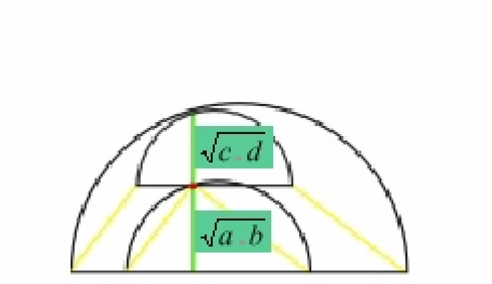


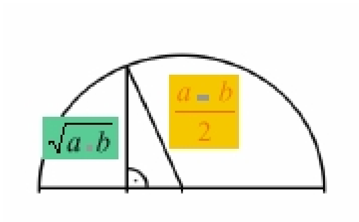



In the *Baker-Smith-problem* the direction of the movements can be well distinguished in a graph. As it is a circle walk, Mr. Smith could also run towards the Bakers, cf. Bruder (2000b). Graphs are also helpful for modelling reflections because they show which assumptions are made or have to be made: constant speed, no breaks, and so on.

Even for the "*root inequality*" visualizations are interesting. However, here again specific basic knowledge is required as for example the visualization of the central inequality which compares the arithmetic with the geometrical center in the semi-circle by means of the Pythagorean theorem in right-angled triangles. A repeated application of the relation of the central inequality for two positive real numbers a and b leads to comprehension.




In this case the algebraic solution would probably be classified as clearer:

$$\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$$

$$ab + ad + cb + cd \overset{?}{\geq} ab + 2\sqrt{abcd} + cd$$

$$ad + cb \overset{?}{\geq} 2\sqrt{abcd}$$

$$a^2d^2 + 2abcd + c^2b^2 \overset{?}{\geq} 4abcd$$

$$(ad - cb)^2 \geq 0$$

This context – averages – also applies to the very first motion problem "*average speed*" where equations are recommended for a solution. In this exercise the harmonic average can be picked up as central theme – also with subsequent visualization.

For the time at constant speed it is:

$$t = \frac{s}{v}$$

v is the searched speed in km/h.
It is :
time of journey $1^{st}$ half + time of journey $2^{nd}$ half = time total

$$\frac{\frac{s}{2}}{50}+\frac{\frac{s}{2}}{v}=\frac{s}{100} \qquad \frac{s}{100}+\frac{s}{2v}=\frac{s}{100} \qquad t_1 + t_2 = t_{gesamt}$$

Interpretation: the planned time for the whole distance is already consumed after the $1^{st}$ half of the journey!

### 4.3.1. Tables

Tables are useful to support systematic trying, e.g. in meeting point problems (cf. the Baker-Smith-problem), or as a possibility to present complete definitions by cases, they help to find different solutions or to represent classifications (value tables) and finally they are applicable for mixed calculations as a special classification form for objects or situations and their different properties.

| Time | Baker | Smith | Remarks |
|---|---|---|---|
| 3 p.m. | 3 km | 0 | Mr. Smith starts |
| 4 p.m. | 6 km | 5 km | Not yet caught up! |
| 5 p.m. | 9 km | 10 km | Already passed! The Bakers are already at home again! |
| 4.30 p.m. | 7,5 km | 7,5 km | Encounter ! |

Systematic trying is an authorized solution strategy where often the seeds of superior mathematical ideas and approaches like nested intervals and approximations are already appearing.

How tables can help to make the invariance principle accessible is demonstrated by Peter/Winkelmaier (2002).
The compiling of a table, the designation of the lines and columns alone corresponds to a considerable intellectual achievement as it depends on prior structuring of the facts. Orienting assistance can be furnished in the form of heuristic rules.
In mixing problems for example it is useful to structure the facts into phases – into the situation before mixing and after mixing. Then the properties are classified: the absolute quantities of the “mixing partners” and the respective share of the absolute quantity. The shares which are often represented as percentage or proportion must be defined as quantities or variables.

### 4.3. Heuristic principles

### 4.3.1. The invariance principle

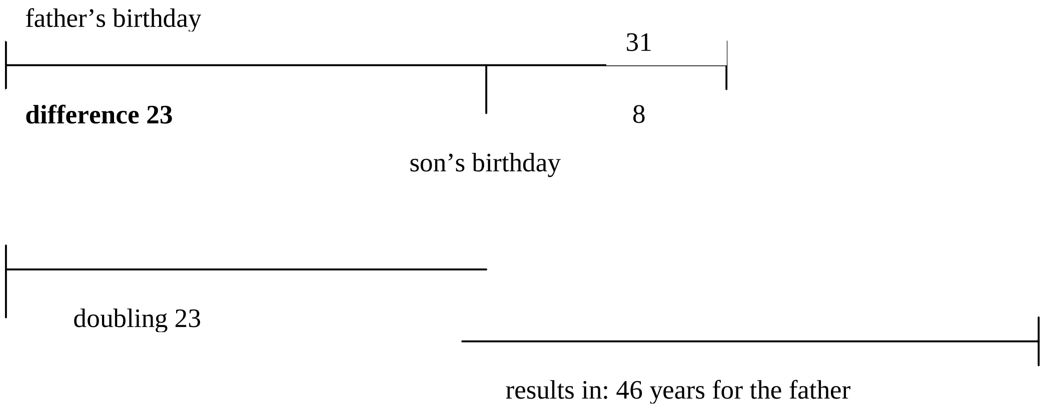
Invariances are sometimes integrated in a problem or have to be constructed.

In the telephone-number-exercise an invariance, corresponding to a difference of 7 between each group of two numbers, has to be found out. This invariance has to be kept up even against obstructions, while the first group of two numbers has simply to be split up. Afterwards it can always be started with a 2 and added a 7 to easily remember both *telephone numbers*.

This simple exercise demonstrates the artificial barriers which are often produced by oneself and in what way the consideration of aspects means intellectual flexibility.

The identification of an age difference by means of an invariant, together with a visualization, can lead to quick solutions of age problems.

For our example "Father and me"  it is:




The construction of an invariance is helpful in exercises of the following category: A pump does "k" l in a given time "t", then it breaks down and another pump with "g" l takes over – and so on. If both pumps are made comparable by a given time unit, asking how many litres each pump does for example per hour or per minute, various filling problems can be solved quickly. Although the artificial practical relevance of such exercises is hardly convincing they furnish thinking experiences for temporal references. The ability to relate objects to situations in time categories is one of the most important mathematisation patterns with respect to their educational value. Further examples on the invariance principle cf. Bruder (1994).

### 4.3.2. The symmetry principle

The "*Root inequality*" is suitable to demonstrate the importance and the idea of such a principle far beyond narrow geometrical conceptions. Again the search for immanent symmetries respectively their reconstruction on the one hand and the destruction or elimination of such symmetries on the other hand are required. In the root inequality the "3" in the numerator on the right and the three "ones" on the left be considered at least as remarkable. Therefore the decomposition of  the 3 into three ones seems the first thing to do, allowing a comparison in pairs of e.g. 1/(a+b) and 1/(a+b+c) which results always in favour of the left side, for positive a,b,c and d.

Let us consider a divisibility problem against the background of the production of symmetry in the sense of a "fitting":

The exercise is:

It is to be shown that $n^3 + 11n$ is divisible by 6 for each natural number $n>0$.

Perhaps this exercise is immediately put aside because a solution by complete induction seems to be the easier way.
This is of course possible but considered as boring and not really elegant. However, heuristic strategies help also to find elegant solutions!

If the symmetry principle is used in the sense that “missing” symmetries are produced as “harmonizing” or “fitting”, a most beautiful solution can be generated:

What is disturbing is the 11. Would there be a 12 instead, something sense-making could be said about the divisibility by 6. Therefore we produce a 12!

The term is now:

$n^3 + 12n – n$

Finally it has to be shown that the term $n^3 – n$ is divisible by 6.
The principle of backward and forward reduction can help:

Forwards: $n^3 - n = n (n^2-1) = n ( n+1) (n-1)$ is a product of three consecutive natural numbers !

Backwards: divisibility by 6 requires divisibility by 2 and 3.

As the product of three consecutive natural numbers is always divisible by 2 and 3 we showed the validity of the statement $6/n^3 + 11n$ for all natural $n>0$.

#### 4.3.3. The extremum principle

Sewerin (1979) describes this proceeding, which is also called selection principle, as the choice of a biggest, smallest or somehow extreme element if the problem deals with the common property of the elements of a finite quantity. In maths lessons there are lots of examples to support this way of thinking:

Example from class 5/6:
For the area A of a square it is 30 cm < A < 36 cm. Find 5 integer measurements for the side lengths of such a square!

Solution idea: provided that the lengths of the sides shall obtain integer measured values, the minimum and maximum dimensions must be determined. The result is a range from a = 1 cm and b = 35 cm to a = 5 cm and b = 7 cm. By means of these pairs of numbers, systematic trying (in a table) can be started. The search for extreme conditions or cases can be useful for typical questions in the maths lessons, such as:

Under which **conditions** does a connection exist...?
or: which minimum requirement (boundary value) must be fulfilled ?

**How many possibilities** are there to fulfil a given requirement?

How many **steps** are at the most / at least necessary to solve a given problem? Or: where are my best **chances**? (games, lottery)

Where is the **optimum**? (minimum material and time input, maximum profit)

Examples:

- *Under which conditions for a, b, and c does the function* $f(x) = ax + bx + c$ *have exactly one zero?*
- *Is the arrangement of Ferrero-Küsschen (sweets) in a cuboid with two layers a perfect packaging?*
- *How can a flowerbed be marked out at maximum size along a house wall with a wire of a certain length?*

The following well-known problem is also a good example. Unfortunately it is often simply taken as application for linear equation systems from class 9:

*In an outdoor-enclosure there is a group of hares and chicken with 12 heads and 36 legs. How many hares and how many chicken are in the enclosure?*

The idea of a minimum condition might lead to a faster solution than the –more current– search for invariables.

**References to the background idea of the extremum principle can be made in the lessons as follows:**

| | |
|---|---|
| Class 5/6 | calculating GCT and LCM<br>finding measures for figures according to given conditions (see above) |
| Class 7/8 | functional reflections, approaches and ideas for an optimization, cf. the herbal garden and sweets packaging examples by Bruder (1997b) |
| Class 9 | introduction to root calculation:<br>*The circumference of a surface* $2m^2$ *?*<br>(square roots are used for circles and squares<br><br>square root calculation by nested intervals – repeated determination of boundary points<br><br>treatment of equation systems in relation with meeting point problems and, more complicated, linear inequality systems |
| Class 11/12 | optimization problems: a volume of 1l is to be packed!<br>Research of the minimum surface of different given bodies |

For the above optimization problem there is an intuitive approach in the SI, which is close to the extremum principle:

Measures are given and varied, value tables are put up.

A reduction of the problem to the surface depending on just one variable will finally be considered as effective. Now it is possible to draw the function curve (with the assistance of graphing calculators), e.g. A=f(r) and to quote a local minimum.
A universal calculation possibility is the derivation method.

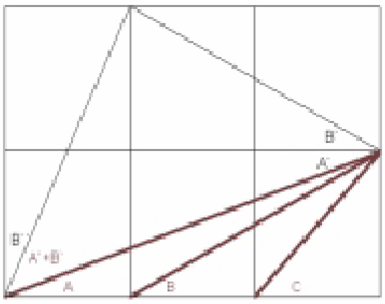
### 4.3.4. The transformation principle

With this principle it is intended to support a change of aspects (cf. manifestations of intellectual flexibility). The following geometry problem offers such a possibility.

Solution by reflection (doubling) of the 3 squares:

The produced isosceles triangles can easily be analysed with respect to the relations between their angles.

It is to be shown that
A+B=C
By supplementing the figure, several isosceles triangles are appearing. By adding the angles it becomes evident that B+A+B+A=2A+2B=90°
A+B=45°=C



Attention:
A=A' and B+B'=B''

## 5. Suggestions on maths lessons with integrated learning of problem solving

Heuristic strategies can only be learned by own experiences unless they are mastered by intuition.

Költzsch (1979) put up the following methodical rules to describe a heuristic learning method:

- Never impose solutions but let students try by themselves, just giving support by gradually narrowing the research area
- Never theorize but introduce each strategy, rule or rule system by carefully chosen examples
- Stimulate input and questions from student to student

The context in which strategies are learned is very important, because strategies are not easily transferable. The examples to support the development of a new strategy must be chosen very carefully as this strategy will remain firmly attached to this introductory context.
However, by giving those introductory examples a pictorial name they can be consciously used as mnemonic to remind of certain proceedings in other contexts and thus support a transfer.

### 5.1 Heurisms can be learned – in four steps

The central idea of our concept is:

Problem solving competences are acquired by forming the intellectual flexibility by training *partial actions* of the problem solving
in connection with *heuristic tools, special principles and general strategies.*

In a long-term learning- and teaching process the following steps and phases must be absolved to reach this target:

**I. Accustoming to heuristic methods and techniques (reflection)**

**II. Becoming conscious of a special method or technique by means of a clear example (forming a strategy)**

**III. Phase of conscious training**

**IV. Expansion of context of the application of strategies**

### I. Accustoming to heuristic methods and techniques

In the first phase the teacher has some kind of a model effect. From primary level heuristically motivated argumentations and questions should be used by intuition in discussions about problems and the different solutions.
Specific heuristic approaches and typical questions are **made used** to the students step by step. In cases of impetus the teacher uses rigorously the question strategies of the different heurisms without making a teaching subject of them.

**Example: principle of analogy**

Teacher's impetus:

- How did we proceed in similar situations?
- What seems well-known to you in this exercise?
- Compare the last exercises (and their solutions) – are there any analogies?

Unlike the learning of a mathematical method -e.g. the solving of linear equation systems- heurisms are learned in four phases over a relatively long time period with short punctual but correlated learning phases.

Heuristics cannot be limited to one teaching unit – they should be always present and accessible to meet their function as effective point of reference in the problem solving.
For the methodical form of phases II and III the following elements proved their worth with respect to students' activities and their necessary own experiences to solve problems:

**II. Becoming conscious of a special method or technique by means of a striking example (forming a strategy)**

The use of heuristic strategies must be learned on the basis of own experiences, cf. the wall brick example in connection with impetus and questions orientated towards forward working.

The strategies which have to be learned explicitly are developed for or presented to each class level by means of **model exercises**. The strategy gets a name and is described by typical questions. The model exercise acts as mnemonic. Together examples are found to show where the strategy, once it is realized, was already adopted earlier by intuition.

The examples stated above can take over such an orientating function.
An important criterion is to find a multitude of solution possibilities for these examples to permit networking as well as a certain transfer to other contexts.

The three heuristic examples *informative figure, table and variable/equation* are especially suited to support the mathematisation process in problems and to reduce a given exercise to essential facts respectively find a better structure.

The presentation of the three heuristic tools in one only suitable model exercise has turned out to be most helpful. In maths-world it is the "bus seat problem" (like the marbles exercise) which can take over this important orientating function, e.g. for classes 7 and 8:

> *In a bus one third of the seats is occupied by children, six seats are occupied by adults. Nine seats are vacant. How many seats are in the bus?*

The fact that this exercise is quite far from reality does not hinder students of this age to think about it, they take it as a game. From class 8 and 9 meeting point problems, which are more related to everyday life, are appropriate (cf. our examples) to experience the three solution variants.

The advantage of this proceeding is the integration of algebraic and geometric knowledge elements and the particularly convincing support of mathematisation capabilities.

Model exercises for the introduction of a heurism should be comprehensible but not too easy in their cognitive requirements to ensure that the additional value of the used heurism is understood.

### III. Phase of conscious training

In the following (short) training phases with exercises of varying difficulty it is expected that the new strategy is applied independently and consciously for optional exercises with different contexts. These optional exercises should clearly differ in their requirements (degree of formalisation, complexity, familiarity) to permit as many individual learning successes as possible.

Individual preferences for certain strategies and the multitude of applications of the new strategy should be discussed as a central theme and thus made conscious.

In this phase it is opportune to train special parts of the problem solving activities separately (developing solution ideas, searching auxiliary lines and quantities). At the end of this phase it should be trained to relate appropriate strategies to problem exercises without solving them. Finally the new heurisms are added to the former individual problem solving model, where the questions method is used (each strategy is described by specific questions).

One ideal way to run this learning phase is an accordingly long-term homework. An example for a one-week homework in class 8 is enclosed.

However, such a proceeding where the choice of the solution way is entirely up to the students, will have consequences also for the written tests. For some exercises the solution way will have to be completely at choice: a solution by systematic trying with a table should then not be less assessed than the solution by an equation or a graphic approach if the requirements of the exercise are fully met.

### IV. Expansion of context of the application of strategies

These rather sporadic phases of training and problem solving are aiming at a gradual, unconscious and **flexible strategy use**. The new strategy obtains a place in the general conceptions how to solve problems.
In accordance with the constructive idea of learning the students will have to elaborate their own problem solving model. They are asked to integrate newly learned heurisms into the existing model by means of own questions and model exercises with a mnemonic function. Typical heuristic principles like reduction, invariance principle, extremum principle and symmetry principle are playing a particular role in the recognition of the multiple relations between the different subjects.

Finally, further reflections and variations/modifications of the exercise will lead to better understanding.

## 5.2. How can problem solving be embedded in a teaching unit?

If we take a large exercise concept which covers the different demands for learning activities, the teaching substance of the lessons can be described as “working with problems”. This includes the construction, choice and formulation of exercises as well as the kind of assistance administered to the students during their processing of the exercises, cf. Bruder (2000d).

**Working with problems** includes

- choosing or constructing, varying, formulating, solving, comparing, evaluating and setting problems by the teacher

- finding, modifying, comparing, setting and solving problems by the students and the assistance in this process by the teacher.

To determinate these multiple activities from the students' point of view, we are talking about **problem processing** in contrast to – what has always been expected most – problem solving.

Working with problems has the following **functions** in the lessons:

- problem processing as **means** (way) to achieve skill and knowledge
- problem processing as **diagnostic instrument** for the proceeding and results in the learning process
- capabilities in problem processing (problem solving!) as intended **objective**

Why should students achieve also abilities in the processing of problems in the sense of the above-mentioned activities? Isn't it sufficient to solve carefully chosen and representative exercises?
If students find, modify and compare exercises, these activities go clearly beyond the solving of current exercises without being automatically much harder. The requirements are a bit different but appropriate to meet the initially mentioned claim for the assumption of responsibilities concerning the own learning and the reflection about the personal proceeding.

The varying of exercises by the students can easily be justified as preparation of a test: Whoever is in a position to transform the formulation of an exercise or to join another aspect to the given problem will be less confused by unfamiliar exercises. Unfortunately it can often be observed that students who are fixed to a certain kind of questions are feeling helpless in front of even smallest modifications of the formulation. Such phenomena will occur especially in case of a teachers change. To gain more flexibility students should learn explicitly how to ask sense-making questions. They should also learn something about the center of interest in the different sciences.

The comparing of problems has particular importance. Actually the problems are meant, not the results! Once a series of several problems or tasks was processed and their results compared or presented, these problems and their solutions are often put aside without taking further activities. In reality this is the phase to gain available skill and knowledge! If the students are asked now to find out common aspects and differences in the just solved problems, the exercises will be analysed once again from another point of view. The given questions can be compared as well as the solution ways and possibly also the results, e.g. with respect to their existence and clearness. The purpose is to grasp the sense of the questions: what was the nature of the problem? Which concepts, proceedings, solution methods and strategies were helpful?

Such working with problems requires prospective working in the important reflection phases as well as a sensible guidance of the students by the teachers in frontal teaching talks. Relevant meta-tasks to compare different problems could be introduced in upper classes or in the form of learning reports or learning diaries which have to be kept independently. There are lots of suitable methods and organisation forms – decisive is above all the quality of the learning requirement.

**Literature**

Ables, L. (2002): Ich hab's – Tipps, Tricks und Übungen zum Problemlösen (I got it – Tips and tricks to solve problems). Mathe-Welt-Beilage in mathematik lehren 115 aus: Wiss. Hausarbeit an der TU Darmstadt 2000: „ Ein Trainingskonzept zur Entwicklung mathematischer Problemlösekompetenzen" (A training concept for the development of problem solving competences)

Baumert and others (1997): TIMSS – Mathematisch-naturwissenschaftlicher Unterricht im internationalen Vergleich. Deskriptive Befunde. (TIMSS – maths teaching in international comparison, descriptive results) Opladen: Leske + Budrich.

Baumert and others (publ.) (2001): PISA 2000.Opladen: Leske+Budrich
Bruder, R., Gürtler, T., Perels, F., Schmitz, B.: Problemlösenlernen in Verbindung mit Selbstregulation (Problem solving learning in connection with self-regulation). In: mathematik lehren (maths teaching)115 (2002), p. 59-62

Bruder, R. (2002): Lernen, geeignete Fragen zu stellen. Heuristik im Mathematikunterricht. (Learning to put suitable questions. Heuristics in maths teaching) In: mathematik lehren 115, p. 4-8

Bruder, R. (2001a): Mathematik lernen und behalten (learning and keeping maths). In: Heymann, H.-W. (publ.): Lernergebnisse sichern (securing learning results). In PÄDAGOGIK, issue 10, p. 15-18

Bruder, R., Weigand, H.-G. (2001): Leistungen bewerten - Natürlich! Aber wie? (Assessing learning results - of course! But how?) In: mathematik lehren 107, p. 4-8

Bruder, R. (2001b): Verständnis für Zahlen, Figuren und Strukturen (Understanding numbers, figures and structures). In: Heymann, H.-W.(publ.): Basiskompetenzen vermitteln (teaching basic competences). In: PÄDAGOGIK, issue 4, p.18-22

Bruder, R. (2000a): Eine akzentuierte Aufgabenauswahl und Vermitteln heuristischer Erfahrung – Wege zu einem anspruchsvollen Mathematikunterricht für alle (An accentuated choice of exercises and teaching of heuristic experience – ways to high-quality maths for everybody).- In: Fladge/Herget (publ.): Mathematik lehren und lernen nach TIMSS – Anregungen für die Sekundarstufen (Teaching and learning maths according to TIMSS – Tips for the secondary level).- Berlin: Volk und Wissen

Bruder, R. (2000b): Problemlösen im Mathematikunterricht – ein Lernangebot für alle? (Problem solving in maths – a teaching purpose for all?) In: Mathematische Unterrichtspraxis (maths practice), issue 1/2000, p.2-11

Bruder, R. (2000c): Konzepte für ein ganzheitliches Unterrichten (concepts for integral teaching). – In: mathematik lehren (maths teaching) 101, p. 4-11

Bruder, R. (2000d): Mit Aufgaben arbeiten (working with exercises).- In: mathematik lehren (maths teaching) 101, p. 12-17

Bruder, R. (1999): Möglichkeiten und Grenzen von Kreativitätsentwicklung im gegenwärtigen Mathematikunterricht (Possibilities and limits of creative development in the present maths teaching). - In: Beiträge zum Mathematikunterricht 1999. Vorträge auf der 33. Bundestagung für Didaktik der Mathematik in Bern (lectures on the 33rd congress for maths didactic in Bern), p. 117-120

Bruder, R.(1997a): Pralinenschachtel, Kinobesuch und fliegende Objekte. Kontext für Binnendifferenzierung (Sweets packaging, cinema visit and flying objects. Context for differentiation)In: mathematik lehren (maths teaching) 89, p. 54-57

Bruder, R. (1997b): Kräutergarten und Konfektverpackung – Optimierung in einer 8.Klasse (Herbary and sweets packaging – optimization in a 8th class). In: mathematik lehren (maths teaching) 89, p. 11-16

Bruder, R. (1994): Im Mathematikunterricht heuristische Erfahrungen gewinnen – am Beispiel des Invarianzbegriffs (Gaining heuristic experiences in maths lessons – with the example of invariance). In: Beiträge zum Lernen und Lehren von Mathematik (Articles for the learning and teaching of maths). Festschrift zur Emeritierung von Martin Glatfeld, Seelze, p. 18-26

Bruder, R. (1993): Verlaufseigenschaften des Denkens im Mathematikunterricht erkennen und fördern (Recognize and support thinking proceedings in maths). In: mathematik lehren (maths teaching), issue 56, p. 20-56

Bruder, R. (1992): Problemlösen – aber wie? (problem solving – but how?) In: mathematik lehren (maths teaching) 52, p. 6-12

Gürtler.,T./Perels,F./Schmitz,B./Bruder,R.: Training zur Förderung selbstregulativer Fähigkeiten in Kombination mit Problemlösen in Mathematik (Training for the development of self-regulative capabilities in connection with mathematical problem solving). – In: Zeitschrift für Pädagogik (pedagogics magazine), 45, special edition 2002, p.222-239

Hasdorf, W.(1976): Erscheinungsbild und Entwicklung der Beweglichkeit des Denkens bei älteren Vorschulkindern (Development of intellectual flexibility of elder preschoolers). In: J. Lompscher: Verlaufsqualitäten der geistigen Tätigkeit (Proceeding of intellectual activity), Berlin: Volk und Wissen

Helmke; A./Jäger, R.S. (2001):; Erster Ergebnisbericht MARKUS (First report on the results MARKUS). Mathematikgesamterhebung Rheinland-Pfalz: Kompetenzen, Unterrichtsmerkmale, Schulkontext (Maths-evaluation in Rhineland-Palatinate: Competences, teaching criteria, school context). Landau: Universität Landau und Zentrum für empirische pädagogische Forschung (www.uni-landau.de/-helmke/) - University of Landau and Center for empiric pedagogic research

Lehmann, G. (1990): Die Vermittlung heuristischer Strategien im Mathematikunterricht (The teaching of heuristic strategies in maths lessons). In:

Finden, Erfinden, Lernen: Zum Umgang mit Mathematik unter heuristischen Aspekten (Finding, inventing, learning: dealing with maths under heuristic aspects), Glatfeld (publ.), Frankfurt a.M.

Lompscher, J. (1972): Theoretische und experimentelle Untersuchungen zur Entwicklung geistiger Fähigkeiten (Theoretic and experimental studies on the development of intellectual capabilities). Berlin: Volk und Wissen

Lompscher, J. (1976): „Verlaufsqualitäten der geistigen Tätigkeit“ (Proceeding of intellectual activity), Berlin: Volk und Wissen

Meyer, Richard E.: “Denken und Problemlösen: eine Einführung in menschliches Denken und Lernen“ (Thinking and prblem solving – an introduction to human thinking and learning), Springer Verlag, Berlin, Heidelberg, New York, 1979

Peter, C., Winkelmaier, C.: Zugang zum Invarianzprinzip über Tabellen (Access to the invariance principle with tables). In: mathematik lehren (maths teaching) 115, p. 10-13

Polya, G. (1949): Schule des Denkens. Vom Lösen mathematischer Probleme. (How to solve it. About mathematical problem solving) Francke: Tübingen und Basel

Polya, G. (1965): Mathematical Discovery (Vol.2). New York: Wiley

Polya, G. (1981): Combined Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving. John Wiley & Sons

Sewerin, H. (1979): Mathematische Schülerwettbewerbe (maths school competitions). München

Winter, H. (1995): Mathematikunterricht und Allgemeinbildung (Maths teaching and general education). In: Mitteilungen der Gesellschaft für Didaktik der Mathematik Nr. 61, p. 37-46